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Thermomechanical postbuckling of shear deformable laminated cylindrical shells with local geometric imperfections

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Abstract

The effect of local geometric imperfections on the buckling and postbuckling of shear deformable laminated cylindrical shells subjected to combined axial compression and uniform temperature loading is investigated. Two cases of compressive postbuckling of initially heated shells and of thermal postbuckling of initially compressed shells are considered. The governing equations are based on Reddy's higher order shear deformation shell theory with a von Kármán–Donnell-type of kinematic nonlinearity and including thermal effects. The material properties are assumed to be independent of the temperature. The nonlinear prebuckling deformations and initial geometric imperfections of the shell are both taken into account. A boundary layer theory of shell buckling is extended to the case of shear deformable cross-ply laminated cylindrical shells and a singular perturbation technique is employed to determine the buckling loads and postbuckling equilibrium paths. The numerical illustrations concern the compressive or thermal postbuckling behavior of moderately thick, cross-ply laminated cylindrical shells with local or modal geometric imperfections. The results show that, for the same value of amplitude, the local geometric imperfection has a small effect on the buckling load as well as postubuckling response of the shell than a modal imperfection does.

Keywords: Thermomechanical postbuckling; Moderately thick laminated cylindrical shell; Local geometric imperfection; Boundary layer theory of shell buckling; Singular perturbation technique

1. Introduction

The postbuckling response of multilayered composite cylindrical shells subjected to combined axial and thermal loads is of current interest to engineers engaged in the aerospace, nuclear, petrochemical and other engineering industries. These cylindrical shells may have significant and unavoidable initial geometric imperfections. Although imperfection distributions are likely to random in the nature, it is often observed that local dimples or modal imperfections are presented in the shell structure. Thus, understanding the effect of initial geometric imperfections on the postbuckling behavior of cylindrical shells is an important consideration in the design of these shells.

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Many postbuckling studies, based on classical shell theory, of composite laminated thin cylindrical shells subjected to mechanical or thermal loading are available in the literature (see for example Sheinman et al., 1983; Shen, 1997b). Relatively few studies involving the application of shear deformation shell theory to postbuckling analysis can be found in Iu and Chia (1988), Reddy and Savoia (1992), Eslami et al. (1998), and Eslami and Shariyat (1999). In the foregoing studies the initial geometric imperfection was assumed to be a modal shaped. However, studies involving the effect of local geometric imperfection on the buckling of cylindrical shells are limited in number. Among those, Hutchinson et al. (1971) gave a number of theoretical and experimental results for the compressive buckling of axially loaded cylindrical shells with a cosine dimple imperfection. Amazigo and Budiansky (1972) gave an imperfection sensitivity analysis of axially compressed cylindrical shells with localized axisymmetric imperfections using Koiter's general theory. The effect of large diamond shaped dimples on the buckling of cylindrical shells under axial compression was investigated experimentally by Krishnakumar and Forster (1991). The influence of a localized defect in the axial direction on the buckling of a long cylindrical shell under axial compression was analyzed by Jamal et al. (1999). In these studies only isotropic cylindrical thin shells were performed and the prebuckling stress state was assumed to be that corresponding to a pure membrane state. Recently, Shen (1999) gave a full nonlinear postbuckling analysis of composite laminated cylindrical shells with local geometric imperfections subjected to axial and thermal loads. It should be noted that in Shen (1999) the shell is considered as being relatively thin and therefore the transverse shear deformation is usually not accounted for.

It has been shown in Shen (1997b) that in shell thermal buckling as well as in shell compressive buckling, there is a boundary layer phenomenon where prebuckling and buckling displacement vary rapidly. This phenomenon was previously reported by Bushnell and Smith (1971). Shen and Chen (1988, 1990) suggested a boundary layer theory of shell buckling, which includes the effects of nonlinear prebuckling deformations, large deflections in the postbuckling range, and initial geometric imperfections of the shell. Based on this theory, the postbuckling analyses for perfect and imperfect, unstiffened and stiffened, laminated cylindrical shells under combined mechanical and thermal loads have been performed by Shen (1997a–c, 1998, 1999). The present study extends the previous work to the case of moderately thick laminated cylindrical shells with local geometric imperfections under combined axial and thermal loads.

In the present study, the temperature field considered is assumed to be a uniform distribution over the shell surface and through the shell thickness. The material properties are assumed to be independent of the temperature. The governing equations are based on Reddy's higher order shear deformation shell theory with a von Kármán–Donnell-type of kinematic nonlinearity and including thermal effects. The nonlinear prebuckling deformations and initial geometric imperfections of the shell are both taken into account. A singular perturbation technique is employed to determine the buckling loads and postbuckling equilibrium paths. The numerical illustrations show the full nonlinear response of shear deformable cross-ply laminated cylindrical shells with local geometric imperfections under two different kinds of loading conditions.

2. Theoretical development

Consider a cylindrical shell with mean radius R, length L and thickness t, which consists of N plies, subjected to two loads combined out of axial compression P_0 and a uniform temperature rise T_0. The shell is referred to a coordinate system (X, Y, Z), in which X and Y are in the axial and circumferential directions of the shell and Z is in the direction of the inward normal to the middle surface, and the corresponding displacement designated by $\overline{U}$, $\overline{V}$ and $\overline{W}$. $\overline{\Psi}_x$ and $\overline{\Psi}_y$ are the rotations of normals to the middle surface with respect to the Y- and X-axes, respectively. The shell is assumed to be relatively thick, geometrically imperfect. Denoting the initial geometric imperfection by $\overline{W}^*(X, Y)$, let $\overline{W}(X, Y)$ be the additional deflection and $\overline{F}(X, Y)$ be the stress function for the stress resultants defined by $\overline{N}_x = \overline{F}_{,yy}$, $\overline{N}_y = \overline{F}_{,xx}$ and $\overline{N}_{xy} = -\overline{F}_{,xy}$, where a comma denotes partial differentiation with respect to the corresponding coordinates.

Reddy and Liu (1985) developed a simple higher order shear deformation shell theory, in which the transverse shear strains are assumed to be parabolically distributed across the shell thickness and which contains the same dependent unknowns as in the first order shear deformation theory. Based on Reddy's higher order shear deformation theory with von Kármán–Donnell-type kinematic relations and including thermal effects, governing differential equations are derived and can be expressed in terms of a stress function $\overline{F}$, two rotations $\overline{\Psi}_x$ and $\overline{\Psi}_y$, and a transverse displacement $\overline{W}$, along with the initial geometric imperfection $\overline{W}^*$. For moderately thick cross-ply laminated cylindrical shells, they are

$$\widetilde{L}_{11}(\overline{W}) - \widetilde{L}_{12}(\overline{\Psi}_x) - \widetilde{L}_{13}(\overline{\Psi}_y) + \widetilde{L}_{14}(\overline{F}) - \widetilde{L}_{15}(\overline{N}^T) - \widetilde{L}_{16}(\overline{M}^T) - \frac{1}{R}\overline{F}_{,xx} = \widetilde{L}(\overline{W} + \overline{W}^*, \overline{F}) \tag{1}$$

$$\widetilde{L}_{21}(\overline{F}) + \widetilde{L}_{22}(\overline{\Psi}_x) + \widetilde{L}_{23}(\overline{\Psi}_y) - \widetilde{L}_{24}(\overline{W}) - \widetilde{L}_{25}(\overline{N}^T) + \frac{1}{R}\overline{W}_{,xx} = -\frac{1}{2}\widetilde{L}(\overline{W} + 2\overline{W}^*, \overline{W}) \tag{2}$$

$$\widetilde{L}_{31}(\overline{W}) + \widetilde{L}_{32}(\overline{\Psi}_x) - \widetilde{L}_{33}(\overline{\Psi}_y) + \widetilde{L}_{34}(\overline{F}) - \widetilde{L}_{35}(\overline{N}^T) - \widetilde{L}_{36}(\overline{S}^T) = 0 \tag{3}$$

$$\widetilde{L}_{41}(\overline{W}) - \widetilde{L}_{42}(\overline{\Psi}_x) + \widetilde{L}_{43}(\overline{\Psi}_y) + \widetilde{L}_{44}(\overline{F}) - \widetilde{L}_{45}(\overline{N}^T) - \widetilde{L}_{46}(\overline{S}^T) = 0 \tag{4}$$

where

$$\begin{aligned}
\tilde{L}_{15}\left(\overline{N}^T\right) &= \left(B^*_{11}\frac{\partial^2}{\partial X^2} + B^*_{12}\frac{\partial^2}{\partial Y^2}\right)\left(\overline{N}^T_x\right) + 2B^*_{66}\frac{\partial^2}{\partial X\partial Y}\left(\overline{N}^T_{xy}\right) + \left(B^*_{21}\frac{\partial^2}{\partial X^2} + B^*_{22}\frac{\partial^2}{\partial Y^2}\right)\left(\overline{N}^T_y\right)\\
\tilde{L}_{16}\left(\overline{M}^T\right) &= \frac{\partial^2}{\partial X^2}\left(\overline{M}^T_x\right) + 2\frac{\partial^2}{\partial X\partial Y}\left(\overline{M}^T_{xy}\right) + \frac{\partial^2}{\partial Y^2}\left(\overline{M}^T_y\right)\\
\tilde{L}_{25}\left(\overline{N}^T\right) &= \left(A^*_{12}\frac{\partial^2}{\partial X^2} + A^*_{11}\frac{\partial^2}{\partial Y^2}\right)\left(\overline{N}^T_x\right) - A^*_{66}\frac{\partial^2}{\partial X\partial Y}\left(\overline{N}^T_{xy}\right) + \left(A^*_{22}\frac{\partial^2}{\partial X^2} + A^*_{12}\frac{\partial^2}{\partial Y^2}\right)\left(\overline{N}^T_y\right)\\
\tilde{L}_{35}\left(\overline{N}^T\right) &= \frac{\partial}{\partial X}\left[\left(B^*_{11} - \frac{4}{3t^2}E^*_{11}\right)\overline{N}^T_x + \left(B^*_{21} - \frac{4}{3t^2}E^*_{21}\right)\overline{N}^T_y\right] + \frac{\partial}{\partial Y}\left[\left(B^*_{66} - \frac{4}{3t^2}E^*_{66}\right)\overline{N}^T_{xy}\right]\\
\tilde{L}_{36}\left(\bar{S}^T\right) &= \frac{\partial}{\partial X}\left(\bar{S}^T_x\right) + \frac{\partial}{\partial Y}\left(\bar{S}^T_{xy}\right)\\
\tilde{L}_{45}\left(\overline{N}^T\right) &= \frac{\partial}{\partial X}\left[\left(B^*_{66} - \frac{4}{3t^2}E^*_{66}\right)\overline{N}^T_{xy}\right] + \frac{\partial}{\partial Y}\left[\left(B^*_{12} - \frac{4}{3t^2}E^*_{12}\right)\overline{N}^T_x + \left(B^*_{22} - \frac{4}{3t^2}E^*_{22}\right)\overline{N}^T_y\right]\\
\tilde{L}_{46}\left(\bar{S}^T\right) &= \frac{\partial}{\partial X}\left(\bar{S}^T_{xy}\right) + \frac{\partial}{\partial Y}\left(\bar{S}^T_y\right)
\end{aligned} \tag{5}$$

and all other linear operators $\widetilde{L}_{ij}(\ )$ and nonlinear operator $\widetilde{L}(\ )$ are defined as in Shen (2001).

It is noted that these shell equations show thermal coupling as well as the interaction of stretching and bending.

Two loading cases are considered. In the first case, a uniform temperature rise is complemented by increasing mechanical compressive edge loading. In the second case, mechanical compressive loading is kept at a constant prebuckling level and the ends of the shell are assumed to be restrained against expansion longitudinally while the uniform temperature is increased steadily. As a result, the boundary conditions are $X = \pm L/2$:

$$\overline{W} = \overline{\Psi}_y = 0, \qquad \overline{M}_x = \overline{P}_x = 0 \quad \text{(simply supported)} \tag{6a}$$

$$\overline{W} = \overline{\Psi}_x = \overline{\Psi}_y = 0 \quad \text{(clamped)} \tag{6b}$$

$$\int_0^{2\pi R} \overline{N}_x \, \mathrm{d}Y + P_0 = 0 \quad \text{(for compressive buckling problem)} \tag{6c}$$

$$\overline{U} = 0 \quad \text{(for thermal buckling problem)} \tag{6d}$$

where $\overline{M}_x$ is the bending moment and $\overline{P}_x$ is higher order moment as defined in Reddy and Liu (1985). Also, we have the closed (or periodicity) condition

$$\int_0^{2\pi R} \frac{\partial \overline{V}}{\partial Y}\,\mathrm{d}Y = 0 \tag{7a}$$

or

$$\int_0^{2\pi R} \left[A_{22}^* \frac{\partial^2 \overline{F}}{\partial X^2} + A_{12}^* \frac{\partial^2 \overline{F}}{\partial Y^2} + \left(B_{21}^* - \frac{4}{3t^2}E_{21}^*\right)\frac{\partial \overline{\Psi}_x}{\partial X} + \left(B_{22}^* - \frac{4}{3t^2}E_{22}^*\right)\frac{\partial \overline{\Psi}_y}{\partial Y} - \frac{4}{3t^2}\left(E_{21}^* \frac{\partial^2 \overline{W}}{\partial X^2} + E_{22}^* \frac{\partial^2 \overline{W}}{\partial Y^2}\right) + \frac{\overline{W}}{R} - \frac{1}{2}\left(\frac{\partial \overline{W}}{\partial Y}\right)^2 - \frac{\partial \overline{W}}{\partial Y}\frac{\partial \overline{W}^*}{\partial Y} - \left(A_{12}^*\overline{N}_x^T + A_{22}^*\overline{N}_y^T\right)\right]\mathrm{d}Y = 0 \tag{7b}$$

Because of Eqs. (7a) and (7b), the in-plane boundary condition $\overline{V} = 0$ (at $X = \pm L/2$) is not needed in Eqs. (6a)–(6d).

The average end-shortening relationship is

$$\begin{aligned} \frac{\Delta_x}{L} &= -\frac{1}{2\pi RL}\int_0^{2\pi R}\int_{-L/2}^{+L/2} \frac{\partial \overline{U}}{\partial X}\,\mathrm{d}X\,\mathrm{d}Y \\ &= -\frac{1}{2\pi RL}\int_0^{2\pi R}\int_{-L/2}^{+L/2} \left[A_{11}^* \frac{\partial^2 \overline{F}}{\partial Y^2} + A_{12}^* \frac{\partial^2 \overline{F}}{\partial X^2} + \left(B_{11}^* - \frac{4}{3t^2}E_{11}^*\right)\frac{\partial \overline{\Psi}_x}{\partial X} + \left(B_{12}^* - \frac{4}{3t^2}E_{12}^*\right)\frac{\partial \overline{\Psi}_y}{\partial Y} \right. \\ &\quad \left. - \frac{4}{3t^2}\left(E_{11}^* \frac{\partial^2 \overline{W}}{\partial X^2} + E_{12}^* \frac{\partial^2 \overline{W}}{\partial Y^2}\right) - \frac{1}{2}\left(\frac{\partial \overline{W}}{\partial X}\right)^2 - \frac{\partial \overline{W}}{\partial X}\frac{\partial \overline{W}^*}{\partial X} - \left(A_{11}^*\overline{N}_x^T + A_{12}^*\overline{N}_y^T\right)\right]\mathrm{d}X\,\mathrm{d}Y \end{aligned} \tag{8}$$

In the above equations and what follows, $[A_{ij}^*]$, $[B_{ij}^*]$, $[D_{ij}^*]$, $[E_{ij}^*]$, $[F_{ij}^*]$ and $[H_{ij}^*]$ ($i, j = 1, 2, 6$) are reduced stiffness matrices, defined by

$$\mathbf{A}^* = \mathbf{A}^{-1}, \quad \mathbf{B}^* = -\mathbf{A}^{-1}\mathbf{B}, \quad \mathbf{D}^* = \mathbf{D} - \mathbf{B}\mathbf{A}^{-1}\mathbf{B}, \quad \mathbf{E}^* = -\mathbf{A}^{-1}\mathbf{E}, \quad \mathbf{F}^* = \mathbf{F} - \mathbf{E}\mathbf{A}^{-1}\mathbf{B}, \quad \mathbf{H}^* = \mathbf{H} - \mathbf{E}\mathbf{A}^{-1}\mathbf{E} \tag{9}$$

where A_{ij}, B_{ij} etc., are the shell stiffnesses, defined in the standard way (see Reddy and Liu, 1985).

The thermal forces, moments and higher order moments caused by temperature rise T_0 are defined by

$$\begin{bmatrix} \overline{N}_x^T & \overline{M}_x^T & \overline{P}_x^T \\ \overline{N}_y^T & \overline{M}_y^T & \overline{P}_y^T \\ \overline{N}_{xy}^T & \overline{M}_{xy}^T & \overline{P}_{xy}^T \end{bmatrix} = \sum_{k=1}^{N} \int_{t_{k-1}}^{t_k} (1, Z, Z^3) \begin{bmatrix} A_x \\ A_y \\ A_{xy} \end{bmatrix}_k T_0\,\mathrm{d}Z \tag{10a}$$

and

$$\begin{bmatrix} \overline{S}_x^T \\ \overline{S}_y^T \\ \overline{S}_{xy}^T \end{bmatrix} = \begin{bmatrix} \overline{M}_x^T \\ \overline{M}_y^T \\ \overline{M}_{xy}^T \end{bmatrix} - \frac{4}{3t^2} \begin{bmatrix} \overline{P}_x^T \\ \overline{P}_y^T \\ \overline{P}_{xy}^T \end{bmatrix} \tag{10b}$$

where

$$\begin{bmatrix} A_x \\ A_y \\ A_{xy} \end{bmatrix} = -\begin{bmatrix} \overline{Q}_{11} & \overline{Q}_{12} & \overline{Q}_{16} \\ \overline{Q}_{12} & \overline{Q}_{22} & \overline{Q}_{26} \\ \overline{Q}_{16} & \overline{Q}_{26} & \overline{Q}_{66} \end{bmatrix} \begin{bmatrix} c^2 & s^2 \\ s^2 & c^2 \\ 2cs & -2cs \end{bmatrix} \begin{bmatrix} \alpha_{11} \\ \alpha_{22} \end{bmatrix} \tag{11}$$

and α_{11} and α_{22} are the thermal expansion coefficients measured in the fiber and transverse directions, respectively, and $\overline{Q}_{ij}$ are the transformed elastic constants, defined in the standard way (see Shen, 1999).

From Eqs. (10a), (10b) and (11) the thermal force $\overline{N}_{xy}^T$, the thermal moments $\overline{M}^T$, and the higher order moments $\overline{P}^T$ are all zero valued, and $\overline{N}_x^T$ and $\overline{N}_y^T$ are both constants, so that $\widetilde{L}_{15}(\overline{N}^T) = \widetilde{L}_{25}(\overline{N}^T) = \widetilde{L}_{35}(\overline{N}^T) = \widetilde{L}_{45}(\overline{N}^T) = \widetilde{L}_{16}(\overline{M}^T) = \widetilde{L}_{36}(\overline{S}^T) = \widetilde{L}_{46}(\overline{S}^T) = 0$.

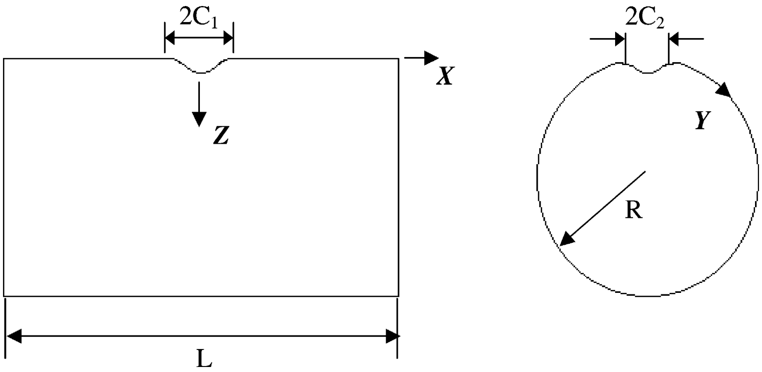

Fig. 1. A cylindrical shell with a local geometric imperfection.

A local asymmetric imperfection is to be assumed as (see Fig. 1)

$$\overline{W}^*(X,Y) = A_m \exp\left(-\left|\frac{X}{C_1}\right| - \left|\frac{Y}{C_2}\right|\right) \tag{12}$$

where A_m is a small parameter characterizing the amplitude of the initial imperfection and C_1 and C_2 characterize the half-width of the region of the dimple. Thus, local means here that the initial deflection decay exponentially in both X- and Y-directions.

3. Analytical method and asymptotic solutions

Having developed the theory, we now try to solve Eqs. (1)–(4) with boundary conditions (6). Before proceeding, it is convenient first to define the following dimensionless quantities (with γ_{ijk} in Eqs. (20), (22) and (23) below are defined as in Shen (2001))

$$\begin{aligned}
&x = \pi X/L, \quad y = Y/R, \quad \beta = L/\pi R, \quad \overline{Z} = L^2/Rt, \quad \varepsilon = \left(\pi^2 R/L^2\right)\left[D_{11}^* D_{22}^* A_{11}^* A_{22}^*\right]^{1/4} \\
&(W, W^*) = \varepsilon\left(\overline{W}, \overline{W}^*\right)/\left[D_{11}^* D_{22}^* A_{11}^* A_{22}^*\right]^{1/4}, \quad F = \varepsilon^2 \overline{F}/\left[D_{11}^* D_{22}^*\right]^{1/2} \\
&\left(\Psi_x, \Psi_y\right) = \varepsilon^2\left(\overline{\Psi}_x, \overline{\Psi}_y\right)(L/\pi)/\left[D_{11}^* D_{22}^* A_{11}^* A_{22}^*\right]^{1/4} \\
&\gamma_{14} = \left[D_{22}^*/D_{11}^*\right]^{1/2}, \quad \gamma_{24} = \left[A_{11}^*/A_{22}^*\right]^{1/2}, \quad \gamma_5 = -A_{12}^*/A_{22}^* \\
&(\gamma_{31}, \gamma_{41}) = \left(L^2/\pi^2\right)\left(A_{55} - 8D_{55}/t^2 + 16F_{55}/t^4, \quad A_{44} - 8D_{44}/t^2 + 16F_{44}/t^4\right)/D_{11}^* \\
&(\gamma_{T1}, \gamma_{T2}) = \left(A_x^T, A_y^T\right)R\Big/\alpha_0\left[D_{11}^* D_{22}^* A_{11}^* A_{22}^*\right]^{1/4}, \quad (\gamma_{C1}, \gamma_{C2}) = (L/\pi C_1, R/C_2) \\
&(M_x, P_x) = \varepsilon^2\left(\overline{M}_x, 4\overline{P}_x/3t^2\right)L^2/\pi^2 D_{11}^*\left[D_{11}^* D_{22}^* A_{11}^* A_{22}^*\right]^{1/4}, \quad \left(\lambda_T, \lambda_T^*\right) = \left(1, 10^3\right)\alpha_0 T_0 \\
&\lambda_p = P_0/4\pi\left[D_{11}^* D_{22}^* A_{11}^* A_{22}^*\right]^{1/4}, \quad \delta_p = (\Delta_x/L)/(2/R)\left[D_{11}^* D_{22}^* A_{11}^* A_{22}^*\right]^{1/4} \\
&\lambda_p^* = P_0[3(1-\nu_{12}\nu_{21})]^{1/2}/2\pi t^2[E_{11}E_{22}]^{1/2}, \quad \delta_p^* = (\Delta_x/L)[3(1-\nu_{12}\nu_{21})]^{1/2}/(t/R)
\end{aligned} \tag{13}$$

and let the thermal expansion coefficients for each ply be

$$\alpha_{11} = a_{11}\alpha_0, \quad \alpha_{22} = a_{22}\alpha_0 \tag{14}$$

where α_0 is an arbitrary reference value, and let

$$\left(A_x^T, A_y^T\right) = -\sum_{k=1}^{N} \int_{t_{k-1}}^{t_k} \left(A_x, A_y\right)_k \mathrm{d}Z \tag{15}$$

The nonlinear Eqs. (1)–(4) may then be written in dimensionless form as

$$\varepsilon^2 L_{11}(W) - \varepsilon L_{12}(\Psi_x) - \varepsilon L_{13}(\Psi_y) + \varepsilon\gamma_{14} L_{14}(F) - \gamma_{14} F_{,xx} = \gamma_{14}\beta^2 L(W + W^*, F) \tag{16}$$

$$L_{21}(F) + \gamma_{24} L_{22}(\Psi_x) + \gamma_{24} L_{23}(\Psi_y) - \varepsilon\gamma_{24} L_{24}(W) + \gamma_{24} W_{,xx} = -\tfrac{1}{2}\gamma_{24}\beta^2 L(W + 2W^*, W) \tag{17}$$

$$\varepsilon L_{31}(W) + L_{32}(\Psi_x) - L_{33}(\Psi_y) + \gamma_{14} L_{34}(F) = 0 \tag{18}$$

$$\varepsilon L_{41}(W) - L_{42}(\Psi_x) + L_{43}(\Psi_y) + \gamma_{14} L_{44}(F) = 0 \tag{19}$$

where

$$\begin{aligned}
L_{11}(\;) &= \gamma_{110}\frac{\partial^4}{\partial x^4} + 2\gamma_{112}\beta^2\frac{\partial^4}{\partial x^2\,\partial y^2} + \gamma_{114}\beta^4\frac{\partial^4}{\partial y^4}\\
L_{12}(\;) &= \gamma_{120}\frac{\partial^3}{\partial x^3} + \gamma_{122}\beta^2\frac{\partial^3}{\partial x\,\partial y^2}\\
L_{13}(\;) &= \gamma_{131}\beta\frac{\partial^3}{\partial x^2\,\partial y} + \gamma_{133}\beta^3\frac{\partial^3}{\partial y^3}\\
L_{14}(\;) &= \gamma_{140}\frac{\partial^4}{\partial x^4} + 2\gamma_{142}\beta^2\frac{\partial^4}{\partial x^2\,\partial y^2} + \gamma_{144}\beta^4\frac{\partial^4}{\partial y^4}\\
L_{21}(\;) &= \frac{\partial^4}{\partial x^4} + 2\gamma_{212}\beta^2\frac{\partial^4}{\partial x^2\,\partial y^2} + \gamma_{214}\beta^4\frac{\partial^4}{\partial y^4}\\
L_{22}(\;) &= \gamma_{220}\frac{\partial^3}{\partial x^3} + \gamma_{222}\beta^2\frac{\partial^3}{\partial x\,\partial y^2}\\
L_{23}(\;) &= \gamma_{231}\beta\frac{\partial^3}{\partial x^2\,\partial y} + \gamma_{233}\beta^3\frac{\partial^3}{\partial y^3}\\
L_{24}(\;) &= \gamma_{240}\frac{\partial^4}{\partial x^4} + 2\gamma_{242}\beta^2\frac{\partial^4}{\partial x^2\,\partial y^2} + \gamma_{244}\beta^4\frac{\partial^4}{\partial y^4}\\
L_{31}(\;) &= \gamma_{31}\frac{\partial}{\partial x} + \gamma_{310}\frac{\partial^3}{\partial x^3} + \gamma_{312}\beta^2\frac{\partial^3}{\partial x\,\partial y^2}\\
L_{32}(\;) &= \gamma_{31} - \gamma_{320}\frac{\partial^2}{\partial x^2} - \gamma_{322}\beta^2\frac{\partial^2}{\partial y^2}\\
L_{33}(\;) &= \gamma_{331}\beta\frac{\partial^2}{\partial x\,\partial y}\\
L_{34}(\;) &= L_{22}(\;)\\
L_{41}(\;) &= \gamma_{41}\beta\frac{\partial}{\partial y} + \gamma_{411}\beta\frac{\partial^3}{\partial x^2\,\partial y} + \gamma_{413}\beta^3\frac{\partial^3}{\partial y^3}\\
L_{42}(\;) &= L_{33}(\;)\\
L_{43}(\;) &= \gamma_{41} - \gamma_{430}\frac{\partial^2}{\partial x^2} - \gamma_{432}\beta^2\frac{\partial^2}{\partial y^2}\\
L_{44}(\;) &= L_{23}(\;)\\
L(\;) &= \frac{\partial^2}{\partial x^2}\frac{\partial^2}{\partial y^2} - 2\frac{\partial^2}{\partial x\,\partial y}\frac{\partial^2}{\partial x\,\partial y} + \frac{\partial^2}{\partial y^2}\frac{\partial^2}{\partial x^2}
\end{aligned} \tag{20}$$

Because of the definition of ε given in Eq. (13), for most of the composite materials $[D_{11}^* D_{22}^* A_{11}^* A_{22}^*]^{1/4} = (0.2 - 0.3)t$, hence when $\overline{Z} = (L^2/Rt) > 2.96$, we have $\varepsilon < 1$. In particular, for isotropic cylindrical shells, we have $\varepsilon = \pi^2/\overline{Z}_B\sqrt{12}$, where $\overline{Z}_B = (L^2/Rt)[1 - \nu^2]^{1/2}$ is the Batdorf shell parameter, which should be greater than 2.85 in the case of classical linear buckling analysis (Batdorf, 1947). In practice, the shell structure will have $\overline{Z} \geqslant 10$, so that we always have $\varepsilon \ll 1$. When $\varepsilon < 1$, Eqs. (16)–(19) are equations of the boundary layer type, from which nonlinear prebuckling deformations, large deflections in the postbuckling range and initial geometric imperfections of the shell can be considered simultaneously.

The boundary conditions of Eqs. (6a)–(6d) become

$x = \pm\pi/2$:

$$W = \Psi_y = 0, \qquad M_x = P_x = 0 \quad \text{(simply supported)} \tag{21a}$$

$$W = \Psi_x = \Psi_y = 0 \quad \text{(clamped)} \tag{21b}$$

$$\frac{1}{2\pi}\int_0^{2\pi} \beta^2 \frac{\partial^2 F}{\partial y^2}\,\mathrm{d}y + 2\lambda_p\varepsilon = 0 \quad \text{(for compressive buckling problem)} \tag{21c}$$

$$\delta_p = 0 \quad \text{(for thermal buckling problem)} \tag{21d}$$

and the closed condition becomes

$$\int_0^{2\pi}\left[\left(\frac{\partial^2 F}{\partial x^2} - \gamma_5\beta^2\frac{\partial^2 F}{\partial y^2}\right) + \gamma_{24}\left(\gamma_{220}\frac{\partial \Psi_x}{\partial x} + \gamma_{522}\beta\frac{\partial \Psi_y}{\partial y}\right) - \varepsilon\gamma_{24}\left(\gamma_{240}\frac{\partial^2 W}{\partial x^2} + \gamma_{622}\beta^2\frac{\partial^2 W}{\partial y^2}\right) + \gamma_{24}W - \frac{1}{2}\gamma_{24}\beta^2\left(\frac{\partial W}{\partial y}\right)^2 - \gamma_{24}\beta^2\frac{\partial W}{\partial y}\frac{\partial W^*}{\partial y} + (\gamma_{T2} - \gamma_5\gamma_{T1})\lambda_T\varepsilon\right]\mathrm{d}y = 0 \tag{22}$$

The unit end-shortening relationship becomes

$$\delta_p = -\frac{1}{4\pi^2\gamma_{24}}\varepsilon^{-1}\int_0^{2\pi}\int_{-\pi/2}^{+\pi/2}\left[\left(\gamma_{24}^2\beta^2\frac{\partial^2 F}{\partial y^2} - \gamma_5\frac{\partial^2 F}{\partial x^2}\right) + \gamma_{24}\left(\gamma_{511}\frac{\partial \Psi_x}{\partial x} + \gamma_{233}\beta\frac{\partial \Psi_y}{\partial y}\right) - \varepsilon\gamma_{24}\left(\gamma_{611}\frac{\partial^2 W}{\partial x^2} + \gamma_{244}\beta^2\frac{\partial^2 W}{\partial y^2}\right) - \frac{1}{2}\gamma_{24}\left(\frac{\partial W}{\partial x}\right)^2 - \gamma_{24}\frac{\partial W}{\partial x}\frac{\partial W^*}{\partial x} + (\gamma_{24}^2\gamma_{T1} - \gamma_5\gamma_{T2})\lambda_T\varepsilon\right]\mathrm{d}x\,\mathrm{d}y \tag{23}$$

By virtue of the fact that T_0 is assumed to be uniform, the thermal coupling in Eqs. (1)–(4) vanishes, but terms in λ_T intervene in Eqs. (22) and (23).

Applying Eqs. (16)–(23), the postbuckling behavior of perfect and imperfect, shear deformable cross-ply laminated cylindrical shells subjected to combined axial and thermal loads is determined by a singular perturbation technique. The essence of this procedure, in the present case, is to assume that

$$\begin{aligned} W &= w(x,y,\varepsilon) + \widetilde{W}(x,\xi,y,\varepsilon) + \widehat{W}(x,\varsigma,y,\varepsilon) \\ F &= f(x,y,\varepsilon) + \widetilde{F}(x,\xi,y,\varepsilon) + \widehat{F}(x,\varsigma,y,\varepsilon) \\ \Psi_x &= \psi_x(x,y,\varepsilon) + \widetilde{\Psi}_x(x,\xi,y,\varepsilon) + \widehat{\Psi}_x(x,\varsigma,y,\varepsilon) \\ \Psi_y &= \psi_y(x,y,\varepsilon) + \widetilde{\Psi}_y(x,\xi,y,\varepsilon) + \widehat{\Psi}_y(x,\varsigma,y,\varepsilon) \end{aligned} \tag{24}$$

where ε is a small perturbation parameter (see beneath Eq. (20)) and $w(x,y,\varepsilon)$, $f(x,y,\varepsilon)$, $\psi_x(x,y,\varepsilon)$ and $\psi_y(x,y,\varepsilon)$ are called outer solutions or regular solutions of the shell, $\widetilde{W}(x,\xi,y,\varepsilon)$, $\widetilde{F}(x,\xi,y,\varepsilon)$, $\widetilde{\Psi}_x(x,\xi,y,\varepsilon)$, $\widetilde{\Psi}_y(x,\xi,y,\varepsilon)$ and $\widehat{W}(x,\varsigma,y,\varepsilon)$, $\widehat{F}(x,\varsigma,y,\varepsilon)$, $\widehat{\Psi}_x(x,\varsigma,y,\varepsilon)$, $\widehat{\Psi}_y(x,\varsigma,y,\varepsilon)$ are the boundary layer solutions near the $x = \pm\pi/2$ edges, respectively, and ξ and ζ are the boundary layer variables, defined by

$$\xi = (\pi/2 + x)/\sqrt{\varepsilon}, \quad \varsigma = (\pi/2 - x)/\sqrt{\varepsilon} \tag{25}$$

(This means for isotropic cylindrical shells the width of the boundary layers is of order $\sqrt{Rt}$). In Eq. (24) the regular and boundary layer solutions are taken in the form of perturbation expansions as

$$\begin{aligned} w(x,y,\varepsilon) &= \sum_{j=1} \varepsilon^{j/2} w_{j/2}(x,y), \quad f(x,y,\varepsilon) = \sum_{j=0} \varepsilon^{j/2} f_{j/2}(x,y) \\ \psi_x(x,y,\varepsilon) &= \sum_{j=1} \varepsilon^{j/2} (\psi_x)_{j/2}(x,y), \quad \psi_y(x,y,\varepsilon) = \sum_{j=1} \varepsilon^{j/2} (\psi_y)_{j/2}(x,y) \end{aligned} \tag{26a}$$

$$\begin{aligned} \widetilde{W}(x,\xi,y,\varepsilon) &= \sum_{j=0} \varepsilon^{j/2+1} \widetilde{W}_{j/2+1}(x,\xi,y), \quad \widetilde{F}(x,\xi,y,\varepsilon) = \sum_{j=0} \varepsilon^{j/2+2} \widetilde{F}_{j/2+2}(x,\xi,y) \\ \widetilde{\Psi}_x(x,\xi,y,\varepsilon) &= \sum_{j=0} \varepsilon^{(j+3)/2} (\widetilde{\Psi}_x)_{(j+3)/2}(x,\xi,y), \quad \widetilde{\Psi}_y(x,\xi,y,\varepsilon) = \sum_{j=0} \varepsilon^{j/2+2} (\widetilde{\Psi}_y)_{j/2+2}(x,\xi,y) \end{aligned} \tag{26b}$$

$$\begin{aligned} \widehat{W}(x,\varsigma,y,\varepsilon) &= \sum_{j=0} \varepsilon^{j/2+1} \widehat{W}_{j/2+1}(x,\varsigma,y), \quad \widehat{F}(x,\varsigma,y,\varepsilon) = \sum_{j=0} \varepsilon^{j/2+2} \widehat{F}_{j/2+2}(x,\varsigma,y) \\ \widehat{\Psi}_x(x,\varsigma,y,\varepsilon) &= \sum_{j=0} \varepsilon^{(j+3)/2} (\widehat{\Psi}_x)_{(j+3)/2}(x,\varsigma,y), \quad \widehat{\Psi}_y(x,\varsigma,y,\varepsilon) = \sum_{j=0} \varepsilon^{j/2+2} (\widehat{\Psi}_y)_{j/2+2}(x,\varsigma,y) \end{aligned} \tag{26c}$$

The initial buckling mode is assumed to have the form

$$w_2(x,y) = A_{11}^{(2)} \cos mxs \cos ny \tag{27}$$

It should be remembered that, because of the definition of W given in Eq. (13), this means that $w_2(x,y)$ corresponds to $\overline{w}_1(X,Y)$ and the initial local geometric imperfection is represented as a Fourier cosine series as

$$\begin{aligned} W^*(x,y,\varepsilon) &= \varepsilon^2 a_m \exp(-\gamma_{C1}|x| - \gamma_{C2}|y|) \\ &= \varepsilon^2 \mu A_{11}^{(2)} \left(\frac{a_0}{2} + \sum_{i=1} a_i \cos ix \right) \left(\frac{b_0}{2} + \sum_{j=1} b_j \cos jy \right) \end{aligned} \tag{28a}$$

where

$$a_i = \frac{4}{\pi} \int_0^{\pi/2} \exp(-\gamma_{C1} x) \cos ix \, dx, \qquad b_j = \frac{2}{\pi} \int_0^{\pi} \exp(-\gamma_{C2} y) \cos jy \, dy \tag{28b}$$

and $\mu = a_m / A_{11}^{(2)}$ is the imperfection parameter.

Substituting Eqs. (24)–(26c) into Eqs. (16)–(19), and collecting terms of the same order of ε, three sets of perturbation equations are obtained for the regular and boundary layer solutions, respectively.

We then use Eqs. (27), (28a) and (28b) to solve these perturbation equations of each order, and match the regular solutions with the boundary layer solutions at each end of the shell, so that the asymptotic solutions satisfying the clamped boundary conditions are constructed as

$$W = \varepsilon\Big[A_{00}^{(1)} - A_{00}^{(1)}\left(a_{01}^{(1)}\cos\phi\frac{\pi/2+x}{\sqrt{\varepsilon}} + a_{10}^{(1)}\sin\phi\frac{\pi/2+x}{\sqrt{\varepsilon}}\right)\exp\left(-\alpha\frac{\pi/2+x}{\sqrt{\varepsilon}}\right)$$
$$- A_{00}^{(1)}\left(a_{01}^{(1)}\cos\phi\frac{\pi/2-x}{\sqrt{\varepsilon}} + a_{10}^{(1)}\sin\phi\frac{\pi/2-x}{\sqrt{\varepsilon}}\right)\exp\left(-\alpha\frac{\pi/2-x}{\sqrt{\varepsilon}}\right)\Big]$$
$$+ \varepsilon^2\Big[A_{11}^{(2)}\cos mx\cos ny + A_{20}^{(2)}\cos 2mx + A_{02}^{(2)}\cos 2ny - \left(-A_{20}^{(2)} + A_{02}^{(2)}\cos 2ny\right)$$
$$\times\left(a_{01}^{(1)}\cos\phi\frac{\pi/2+x}{\sqrt{\varepsilon}} + a_{10}^{(1)}\sin\phi\frac{\pi/2+x}{\sqrt{\varepsilon}}\right)\exp\left(-\alpha\frac{\pi/2+x}{\sqrt{\varepsilon}}\right) - \left(-A_{20}^{(2)} + A_{02}^{(2)}\cos 2ny\right)$$
$$\times\left(a_{01}^{(1)}\cos\phi\frac{\pi/2-x}{\sqrt{\varepsilon}} + a_{10}^{(1)}\sin\phi\frac{\pi/2-x}{\sqrt{\varepsilon}}\right)\exp\left(-\alpha\frac{\pi/2-x}{\sqrt{\varepsilon}}\right)\Big]$$
$$+ \varepsilon^3\left[A_{11}^{(3)}\cos mx\cos ny + A_{02}^{(3)}\cos 2ny\right] + \varepsilon^4\Big[A_{00}^{(4)} + A_{20}^{(4)}\cos 2mx + A_{02}^{(4)}\cos 2ny$$
$$+ A_{13}^{(4)}\cos mx\cos 3ny + A_{04}^{(4)}\cos 4ny\Big] + O(\varepsilon^5) \tag{29}$$

$$F = -\frac{y^2}{2}B_{00}^{(0)} + \varepsilon\left[-\frac{y^2}{2}B_{00}^{(1)}\right] + \varepsilon^2\Big[-\frac{y^2}{2}B_{00}^{(2)} + B_{11}^{(2)}\cos mx\cos ny + A_{00}^{(1)}\Big(b_{01}^{(2)}\cos\phi\frac{\pi/2+x}{\sqrt{\varepsilon}}$$
$$+ b_{10}^{(2)}\sin\phi\frac{\pi/2+x}{\sqrt{\varepsilon}}\Big)\exp\left(-\alpha\frac{\pi/2+x}{\sqrt{\varepsilon}}\right) + A_{00}^{(1)}\left(b_{01}^{(2)}\cos\phi\frac{\pi/2-x}{\sqrt{\varepsilon}} + b_{10}^{(2)}\sin\phi\frac{\pi/2-x}{\sqrt{\varepsilon}}\right)$$
$$\times\exp\left(-\alpha\frac{\pi/2-x}{\sqrt{\varepsilon}}\right)\Big] + \varepsilon^3\Big[-\frac{y^2}{2}B_{00}^{(3)} + B_{02}^{(3)}\cos 2ny + \left(-A_{20}^{(2)} + A_{02}^{(2)}\cos 2ny\right)$$
$$\times\left(b_{01}^{(3)}\cos\phi\frac{\pi/2+x}{\sqrt{\varepsilon}} + b_{10}^{(3)}\sin\phi\frac{\pi/2+x}{\sqrt{\varepsilon}}\right)\exp\left(-\alpha\frac{\pi/2+x}{\sqrt{\varepsilon}}\right) + \left(-A_{20}^{(2)} + A_{02}^{(2)}\cos 2ny\right)$$
$$\times\left(b_{01}^{(3)}\cos\phi\frac{\pi/2-x}{\sqrt{\varepsilon}} + b_{10}^{(3)}\sin\phi\frac{\pi/2-x}{\sqrt{\varepsilon}}\right)\exp\left(-\alpha\frac{\pi/2-x}{\sqrt{\varepsilon}}\right)\Big]$$
$$+ \varepsilon^4\left[-\frac{y^2}{2}B_{00}^{(4)} + B_{11}^{(4)}\cos mx\cos ny + B_{20}^{(4)}\cos 2mx + B_{02}^{(4)}\cos 2ny + B_{13}^{(4)}\cos mx\cos 3ny\right] + O(\varepsilon^5) \tag{30}$$

$$\Psi_x = \varepsilon^{3/2}\left[A_{00}^{(1)}c_{10}^{(3/2)}\sin\phi\frac{\pi/2+x}{\sqrt{\varepsilon}}\exp\left(-\alpha\frac{\pi/2+x}{\sqrt{\varepsilon}}\right) + A_{00}^{(1)}c_{10}^{(3/2)}\sin\phi\frac{\pi/2-x}{\sqrt{\varepsilon}}\exp\left(-\alpha\frac{\pi/2-x}{\sqrt{\varepsilon}}\right)\right]$$
$$+ \varepsilon^2\left[C_{11}^{(2)}\sin mx\cos ny\right] + \varepsilon^{5/2}\Big[(-A_{20}^{(2)} + A_{02}^{(2)}\cos 2ny)c_{10}^{(5/2)}\sin\phi\frac{\pi/2+x}{\sqrt{\varepsilon}}\exp\left(-\alpha\frac{\pi/2+x}{\sqrt{\varepsilon}}\right)$$
$$+ \left(-A_{20}^{(2)} + A_{02}^{(2)}\cos 2ny\right)c_{10}^{(5/2)}\sin\phi\frac{\pi/2-x}{\sqrt{\varepsilon}}\exp\left(-\alpha\frac{\pi/2-x}{\sqrt{\varepsilon}}\right)\Big] + \varepsilon^3\left[C_{11}^{(3)}\sin mx\cos ny\right]$$
$$+ \varepsilon^4\left[C_{11}^{(4)}\sin mx\cos ny + C_{20}^{(4)}\sin 2mx + C_{13}^{(4)}\sin mx\cos 3ny\right] + O(\varepsilon^5) \tag{31}$$

$$\begin{aligned}\Psi_y = {} & \varepsilon^2\left[D_{11}^{(2)}\cos mxc\sin ny\right] + \varepsilon^3\left[D_{11}^{(3)}\cos mx\sin ny + D_{02}^{(3)}\sin 2ny\right.\\ & - \left(A_{02}^{(2)}2n\beta\sin 2ny\right)\left(d_{01}^{(3)}\cos\phi\frac{\pi/2+x}{\sqrt{\varepsilon}} + d_{10}^{(3)}\sin\phi\frac{\pi/2+x}{\sqrt{\varepsilon}}\right)\exp\left(-\alpha\frac{\pi/2+x}{\sqrt{\varepsilon}}\right)\\ & \left.- \left(A_{02}^{(2)}2n\beta\sin 2ny\right)\left(d_{01}^{(3)}\cos\phi\frac{\pi/2-x}{\sqrt{\varepsilon}} + d_{10}^{(3)}\sin\phi\frac{\pi/2-x}{\sqrt{\varepsilon}}\right)\exp\left(-\alpha\frac{\pi/2-x}{\sqrt{\varepsilon}}\right)\right]\\ & + \varepsilon^4\left[D_{11}^{(4)}\cos mx\sin ny + D_{02}^{(4)}\sin 2ny + D_{13}^{(4)}\cos mx\sin 3ny\right] + \mathrm{O}(\varepsilon^5)\end{aligned} \tag{32}$$

Note that, all of the coefficients in Eqs. (29)–(32) are related and can be written as functions of $A_{11}^{(2)}$, but for the sake of brevity the detailed expressions are not shown, whereas α and ϕ are given in detail in Appendix A.

Next, substituting Eqs. (29)–(32) into boundary condition Eq. (21c), and closed condition Eq. (22) and into Eq. (23), the postbuckling equilibrium paths for initially heated shells can be written as

$$\lambda_p = \left(1-\frac{T_0}{T_{\mathrm{cr}}}\right)\lambda_p^{(0)} - \lambda_p^{(1)}\left(A_{11}^{(2)}\varepsilon\right) - \lambda_p^{(2)}\left(A_{11}^{(2)}\varepsilon\right)^2 + \lambda_p^{(3)}\left(A_{11}^{(2)}\varepsilon\right)^3 + \lambda_p^{(4)}\left(A_{11}^{(2)}\varepsilon\right)^4 + \cdots \tag{33}$$

and

$$\delta_p = \delta_p^{(0)} + \delta_p^{(2)}\left(A_{11}^{(2)}\varepsilon\right)^2 + \delta_p^{(4)}\left(A_{11}^{(2)}\varepsilon\right)^4 + \cdots \tag{34}$$

in Eqs. (33) and (34), $\left(A_{11}^{(2)}\varepsilon\right)$ is taken as the second perturbation parameter relating to the dimensionless maximum deflection. If the maximum deflection is assumed to be at the point $(x,y)=(0,0)$, from Eq. (29), one has

$$A_{11}^{(2)}\varepsilon = W_m - \Theta_1 W_m^2 + \cdots \tag{35a}$$

where W_m is the dimensionless form of maximum deflection of the shell, which can be written as

$$W_m = \frac{1}{C_3}\left[\frac{t}{[D_{11}^*D_{22}^*A_{11}^*A_{22}^*]^{1/4}}\frac{\overline{W}}{t} + \Theta_2\right] \tag{35b}$$

All symbols used in Eqs. (33)–(35b) and Eqs. (36), (37a) and (37b) below are also described in detail in Appendix A.

Similarly, substituting Eqs. (29)–(32) into boundary condition Eq. (21d), and closed condition Eq. (22), the thermal postbuckling equilibrium paths for initially compressed shells can be written as

$$\lambda_T = C_{11}\left[\left(1-\frac{P_0}{P_{\mathrm{cr}}}\right)\lambda_T^{(0)} - \lambda_T^{(1)}\left(A_{11}^{(2)}\varepsilon\right) - \lambda_T^{(2)}\left(A_{11}^{(2)}\varepsilon\right)^2 + \lambda_T^{(3)}\left(A_{11}^{(2)}\varepsilon\right)^3 + \lambda_T^{(4)}\left(A_{11}^{(2)}\varepsilon\right)^4 + \cdots\right] \tag{36}$$

in Eq. (36), $\left(A_{11}^{(2)}\varepsilon\right)$ is also taken as the second perturbation parameter in this case, and one has

$$A_{11}^{(2)}\varepsilon = W_m - \Theta_3 W_m^2 + \cdots \tag{37a}$$

and the dimensionless maximum deflection of the shell is written as

$$W_m = \frac{1}{C_3}\left[\frac{t}{[D_{11}^*D_{22}^*A_{11}^*A_{22}^*]^{1/4}}\frac{\overline{W}}{t} + \Theta_4\right] \tag{37b}$$

Eqs. (33)–(37b) can be employed to obtain numerical results for the postbuckling load-shortening or load-deflection curves of moderately thick laminated cylindrical shells with local geometric imperfections subjected to combined axial and thermal loads, specially for the two cases of compressive postbuckling of initially heated laminated shells, and thermal postbuckling of initially compressed laminated shells. Buckling under pure axial compression and buckling under pure uniform temperature rise follow as two limiting cases. It is noted that for most moderately thick laminated cylindrical shells the critical value of temperature rise T_{cr} is very high, and in such a case the thermal buckling due to uniform temperature alone will not occur. In contrast, when an initial compressive load is applied and kept at a high level, such thermal buckling can occur. For this reason, in the next section we take $P_0/P_{cr} = 0.7$ and 0.8 in the numerical analysis for thermal buckling problems. From Appendix A, equations for the critical value of compressive load P_{cr} or temperature rise T_{cr} can easily be found. The initial buckling load of a perfect shell can readily be obtained numerically, by setting $\overline{W}^*/t = 0$ (or $\mu = 0$), while taking $\overline{W}/t = 0$ (note that $W_m \neq 0$). In all cases, the minimum buckling load is determined by considering Eq. (33) or (36) for various values of the buckling mode (m, n), which determine the number of half-waves in the X-direction and of full waves in the Y-direction. Note that because of Eq. (29), the prebuckling deformation of the shell is nonlinear.

4. Numerical results and comments

Numerical results are presented in this section for moderately thick, antisymmetric or symmetric cross-ply laminated cylindrical shells with or without local or modal imperfections, where the outmost layer is the first mentioned orientation. The initial buckling modal imperfection is defined as $W^*(x, y, \varepsilon) = \varepsilon^2 a_m \cos mx \cos ny$, for which the results were obtained numerically in the manner described previously and detailed further in Shen (2002). Typical results are presented in dimensionless graphical forms in which λ_p^* and δ_p^* are used for initially heated shells and λ_T^* is used for initially compressed shells. For these examples (except for Table 1), all plies are of equal thickness and the total thickness of the shell is $t = 0.01$ m; and the local imperfection parameters are $C_1/L = C_2/R = 0.02$; and material properties are: $E_{11} = 155$ GPa, $E_{22} = 8.07$ GPa, $G_{12} = G_{13} = 4.55$ GPa, $G_{23} = 3.25$ GPa, $\nu_{12} = 0.22$, $\alpha_{11}/\alpha_0 = -0.07$, $\alpha_{22}/\alpha_0 = 30.1$ and $\alpha_0 = 10^{-6}$/°C. It should be appreciated that in all of these figures $\overline{W}^*/t$ and $\overline{W}/t$ mean the dimensionless forms of, respectively, the maximum initial geometric imperfection and additional deflection of the shell.

The accuracy and effectiveness of the present method for buckling analysis of shear deformable laminated cylindrical shells under axial compression, excluding temperature effects, were examined by many comparison studies given in Shen (2002), e.g. the buckling loads for $(0/90)_T$ antisymmetric laminated cylindrical shells were compared with the first order shear deformation theory (FSDT) results of Iu and Chia (1988), and for $(0/90/0)_S$ and $(90/0/90)_S$ symmetric laminated cylindrical shells were compared with the first order shear deformation theory (FOST) and higher order shear deformation theory (HOST) results of Anastasiadis et al. (1994), and for single-layer orthotropic cylindrical shells were compared with three-dimensional solutions of Kardomateas (1995). These comparisons show that the results from present method are in good agreement with existing results. In addition, the buckling loads for simply supported, $(0/90)_T$ and (0/90/0) cross-ply laminated cylindrical shells under axial compression are calculated and compared in Table 1 with the results obtained by Khdeir et al. (1989) based on a higher order shear

Table 1
Comparisons of buckling loads $\sigma_x(L/t)^2/100E_{22}$ for perfect cross-ply laminated cylindrical shells under axial compression

Lay-up	Present	Khdeir et al. (1989)		
	HSDT	HSDT	FSDT	CLT
$(0/90)_T$	0.1652	0.1687	0.1670	0.1817
(0/90/0)	0.2782	0.2794	0.2813	0.4186

$E_{11} = 40E_{22}$, $G_{12} = G_{13} = 0.6E_{22}$, $G_{23} = 0.5E_{22}$, $\nu_{12} = 0.25$, and $L/R = 1$, $R/t = 10$.

deformation theory (HSDT) along with the first order shear deformation theory (FSDT) and the classical laminate theory (CLT). The material properties adopted here are $E_{11} = 40\ E_{22}$, $G_{12} = G_{13} = 0.6\ E_{22}$, $G_{23} = 0.5\ E_{22}$ and $v_{12} = 0.25$. It can be seen that the present results agree well but slightly lower than those of Khdeir et al. (1989).

Figs. 2 and 3 give, respectively, the compressive postbuckling load-shortening and load-deflection curves for perfect ($\overline{W}^*/t = 0$) and imperfect ($\overline{W}^*/t = 0.1$), $(0/90)_{2T}$ and $(0/90)_S$ laminated cylindrical shells under different values of the initial thermal loading T_0 shown. It can be seen that a well-known "snap-through" phenomenon occurs in the postbuckling range. The elastic limit load can be achieved for a small imperfection and in such a case imperfection sensitivity can be predicted. It can also be seen that the buckling loads are reduced with increases in temperature, and the postbuckling path becomes significantly lower.

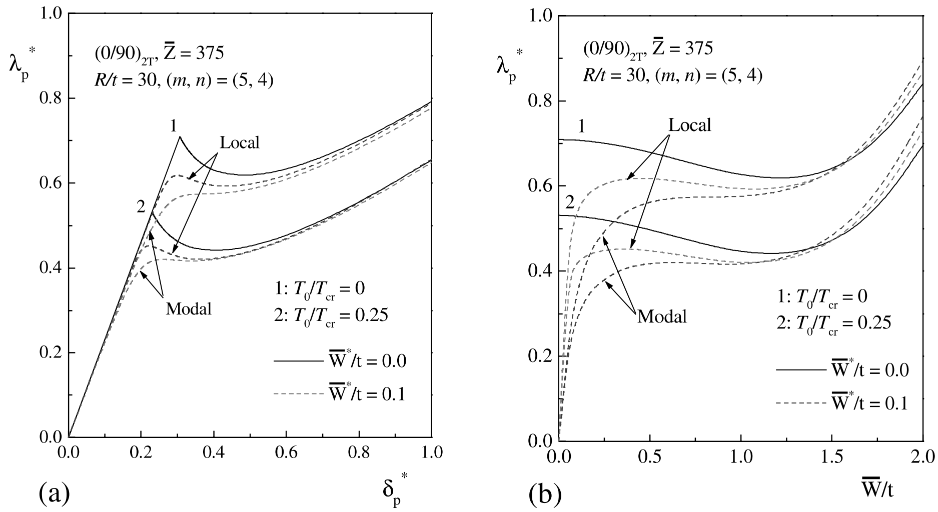

Fig. 2. Postbuckling behavior of an initially heated $(0/90)_{2T}$ laminated cylindrical shell: (a) load-shortening; (b) load-deflection.

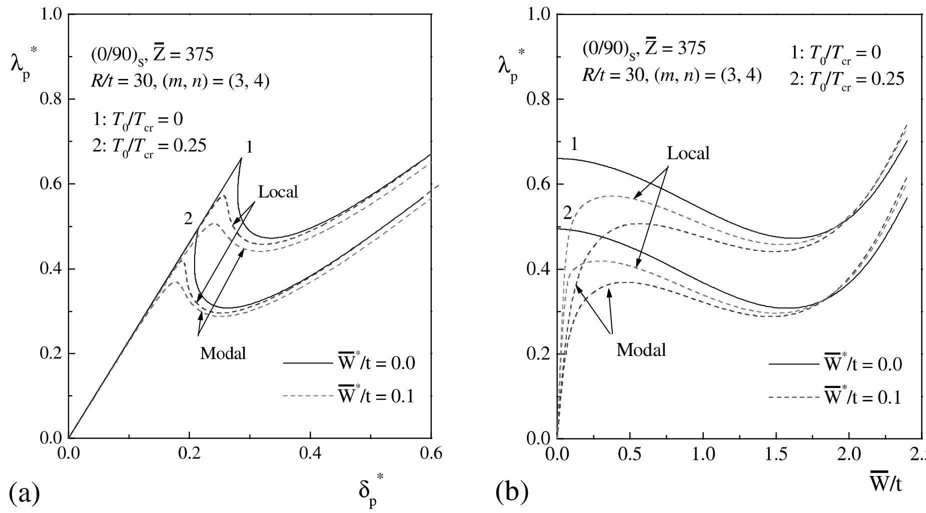

Fig. 3. Postbuckling behavior of an initially heated $(0/90)_S$ laminated cylindrical shell: (a) load-shortening; (b) load-deflection.

The results reveal that the postbuckling load for the shell with a local geometric imperfection is greater than that of the shell with modal imperfection.

Fig. 4 shows curves of imperfection sensitivity for initially heated $(0/90)_{2T}$ and $(0/90)_S$ laminated cylindrical shells with local or modal imperfections. Here, λ^* is the maximum value of σ_x for the imperfect shell, made dimensionless by dividing by the critical value of σ_x for the perfect shell. These results show that the imperfection sensitivity of the $(0/90)_{2T}$ shell is weaker than that of $(0/90)_S$ shell. The imperfection sensitivity of shells with local imperfections is weaker than that of shells with modal imperfections. Also the imperfection sensitivity of an initial heated shell is slightly great than that of the shell without any initial thermal stress. Note that the results presented here are only for a small initial geometric imperfection.

Figs. 5–7 are the thermal postbuckling results for initially compressed shells analogous to the compressive postbuckling results of Figs. 2–4, but without load-shortening curves. As argued earlier, in Figs. 5–7 the initial compressive loads adopted are $P_0/P_{cr} = 0.7$ and 0.8. Fig. 5 shows that if the temperature rise

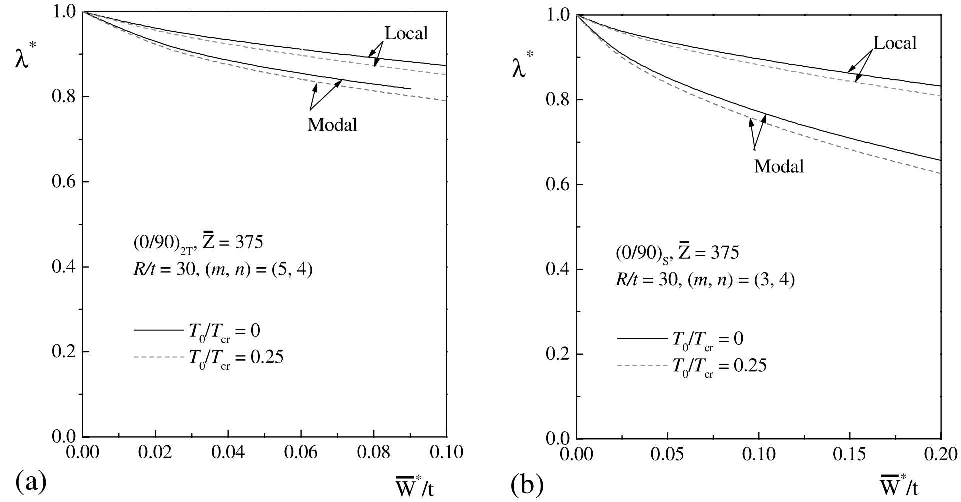

Fig. 4. Comparisons of imperfection sensitivities of initially heated cylindrical shells under axial compression: (a) $(0/90)_{2T}$; (b) $(0/90)_S$.

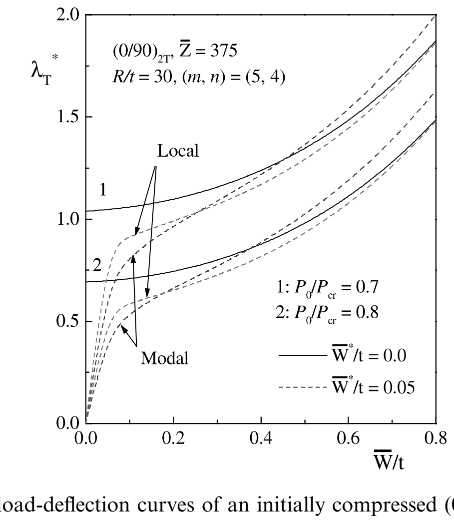

Fig. 5. Thermal postbuckling load-deflection curves of an initially compressed $(0/90)_{2T}$ laminated cylindrical shell.

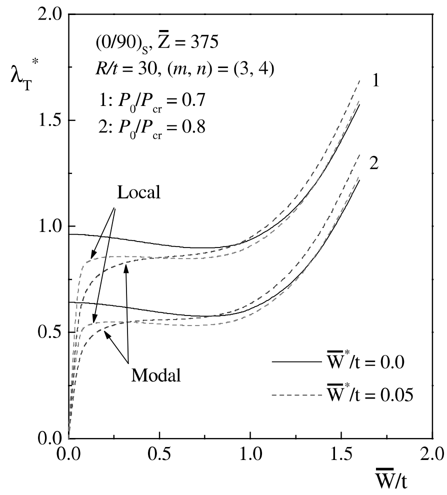

Fig. 6. Thermal postbuckling load-deflection curves of an initially compressed $(0/90)_S$ laminated cylindrical shell.

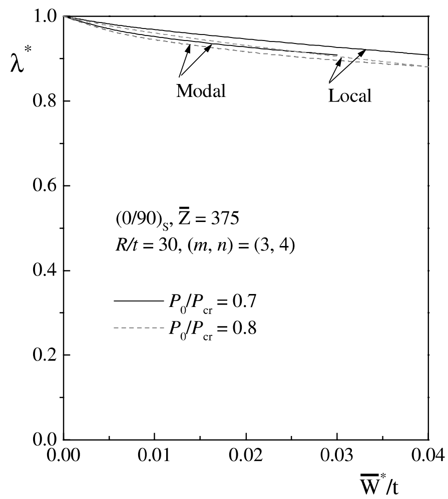

Fig. 7. Comparisons of imperfection sensitivities of an initially compressed $(0/90)_S$ cylindrical shell under uniform temperature loading.

exceeds a critical buckling level, the thermal postbuckling load-deflection curves of an initially compressed $(0/90)_{2T}$ shell go upward dramatically, and the shell structure is virtually imperfection insensitive. Otherwise, Figs. 6 and 7 lead to broadly the same conclusions for the $(0/90)_S$ laminated cylindrical shell as do Figs. 3 and 4. Note that in Fig. 7, now λ^* is the maximum value of λ_T^* for the imperfect shell, made dimensionless by dividing by the critical value of λ_T^* for the perfect shell.

5. Concluding remarks

In order to assess the effect of local geometric imperfections on the postbuckling behavior of shear deformable laminated cylindrical shell subjected to combined axial compression and a uniform temperature rise, a fully nonlinear postbuckling analysis is presented. Two cases of compressive postbuckling of initially heated shells and of thermal postbuckling of initially compressed shells are considered. The material properties are assumed to be independent of the temperature. The boundary layer theory of shell buckling is extended to the case of shear deformable laminated cylindrical shells, and a singular perturbation technique is employed to determine buckling loads and postbuckling equilibrium paths. The numerical examples presented relate to the performance of moderately thick, cross-ply laminated cylindrical shells with local or modal imperfections. The results show that, for the same value of amplitude, the local geometric imperfection has a small effect on the buckling load as well as postbuckling response of the shell than a modal imperfection does. These results also confirm that for moderately thick laminated cylindrical shells the thermal buckling due to uniform temperature alone will not occur. In contrast, for large value of initial compressive loads (e.g. $P_0/P_{cr} = 0.7$ and 0.8), such thermal buckling can occur.

Acknowledgements

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Appendix A

In Eqs. (33)–(35b)

$$\Theta_1 = \frac{1}{C_3}\left[\gamma_{14}\gamma_{24}\frac{m^4\mu_{11}}{16n^2\beta^2 g_{09}g_{06}}\varepsilon^{-1} - \gamma_{14}\gamma_{24}\frac{m^2 g_{11}}{32n^2\beta^2 g_{09}} + \frac{2\gamma_5}{\gamma_{24}}\lambda_p^{(2)}\right]$$

$$\Theta_2 = 2\left[\frac{\gamma_5}{\gamma_{24}}\left(1-\frac{T_0}{T_{cr}}\right) + \frac{\gamma_{24}^2-\gamma_5^2}{\gamma_{24}}\frac{\gamma_{T2}}{g_T}\frac{T_0}{T_{cr}}\right]\lambda_p^{(0)}$$

$$C_3 = 1 - \mu d_{20} - \frac{g_{05}}{m^2}\varepsilon + \frac{2\gamma_5}{\gamma_{24}}\lambda_p^{(1)}$$

$$\lambda_p^{(0)} = \frac{1}{2}\left\{\frac{\gamma_{24}m^2}{\mu_{11}g_{06}}\varepsilon^{-1} + \gamma_{24}\frac{g_{05}+\mu_{11}g_{07}}{\mu_{11}^2 g_{06}} + \frac{1}{\gamma_{14}\mu_{11}m^2}\left[g_{08} + \gamma_{14}\gamma_{24}\frac{g_{05}}{g_{06}}\frac{\mu_{11}g_{07}-\mu d_{11}\mu_{21}g_{05}}{\mu_{11}^2}\right]\varepsilon\right.$$
$$\left. - \frac{\mu_{301}g_{05}}{\gamma_{14}m^4}\left[1+\frac{g_{05}}{\mu_{11}m^2}\varepsilon\right]\left[g_{08} + \gamma_{14}\gamma_{24}\frac{g_{05}}{g_{06}}\frac{g_{05}+\mu_{11}g_{07}}{\mu_{11}^2}\mu_{21}\right]\varepsilon^2\right\}$$

$$\lambda_p^{(1)} = \frac{1}{2}\left\{\frac{4\gamma_{24}m^2n^2\beta^2}{\mu_{11}g_{06}}(\mu d_{02}) + 2\gamma_{24}n^2\beta^2\frac{g_{05}+\mu_{11}g_{07}}{\mu_{11}g_{06}}(\mu d_{02})\varepsilon + \frac{8m^2n^2\beta^2}{\gamma_{14}}\frac{S_0}{S_1}(\mu d_{20})\varepsilon^2\right\}$$

$$\lambda_p^{(2)} = \frac{1}{8}\left\{\gamma_{14}\gamma_{24}^2\frac{m^6\mu_{213}}{2g_{09}g_{06}^2}\varepsilon^{-1} + \gamma_{14}\gamma_{24}\frac{m^4}{2g_{09}g_{06}}\left[\frac{g_{05}}{g_{06}\mu_{11}} + \frac{g_{07}}{g_{06}}\mu_{113} + g_{12}\mu_{11} - \frac{1}{2}\mu_{201}g_{11}\right] - \frac{1}{4}\gamma_{24}m^2 g_{13}2\mu_{222}\varepsilon\right.$$
$$\left. + \gamma_{14}\gamma_{24}^2\frac{m^2 g_{11}}{2g_{09}}\left[\frac{g_{05}}{g_{06}\mu_{11}} - \frac{g_{07}}{g_{06}} - g_{12}\right]\varepsilon + \gamma_{14}\gamma_{24}^2\frac{m^2 g_{05}}{2g_{09}g_{06}}\left[\frac{1}{2}\mu_{420}g_{14} + \mu_{302}\frac{g_{05}}{g_{06}}\right]\mu_{21}\varepsilon + \gamma_{24}\frac{m^2n^4\beta^4}{g_{06}}\frac{S_2}{S_1}\varepsilon\right\}$$

$$\lambda_p^{(3)} = \frac{1}{8}\left\{\gamma_{14}\gamma_{24}^2\frac{m^6n^2\beta^2}{g_{09}g_{06}^2}\frac{g_{136}\mu_{431}+g_{06}\mu_{432}}{g_{136}-g_{06}\mu_{11}}(\mu d_{02}) + 4\gamma_{24}\frac{m^6n^2\beta^2}{g_{06}^2}\frac{g_{136}+g_{06}\mu_{31}}{g_{136}-g_{06}\mu_{11}}\mu_{11}(\mu d_{04})\right\}$$

$$\lambda_p^{(4)} = \frac{1}{128}\gamma_{14}^2\gamma_{24}^3\frac{m^{10}\mu_{11}}{g_{09}^2 g_{06}^3}\frac{S_3}{S_{13}}\varepsilon^{-1}$$

$$\delta_p^{(0)} = \frac{1}{\gamma_{24}}\left[\gamma_{24}^2 - \frac{2}{\pi}\frac{\gamma_5^2}{\gamma_{24}}(\alpha b_{01}^{(2)} - \phi b_{10}^{(2)})\varepsilon^{1/2}\right]\lambda_p + \left[\frac{b_{11}}{2\pi\alpha}\frac{\gamma_5^2}{\gamma_{24}^2}\varepsilon^{1/2}\right]\lambda_p^2$$

$$\delta_p^{(2)} = \frac{1}{16}\left\{m^2\left[\mu_{12} - 8(\mu d_{20})^2\right]\varepsilon - 2g_{05}\varepsilon^2 + \frac{g_{05}^2}{m^2}\varepsilon^3\right\}$$

$$\delta_p^{(4)} = \frac{1}{128}\left\{\frac{b_{11}}{32\pi\alpha}\gamma_{14}^2\gamma_{24}^2\frac{m^8\mu_{11}^2}{n^4\beta^4 g_{09}^2 g_{06}^2}\varepsilon^{-3/2} + m^2n^4\beta^4\mu_{11}^2\left(\frac{S_4}{S_1}\right)^2\varepsilon^3\right\} \tag{A.1}$$

$$S_0 = g_{06}\mu_{11} + 4m^2\mu_{21}g_{10}, \quad S_1 = g_{06}\mu_{11} - 4m^2 g_{10}$$

$$S_2 = g_{06}\mu_{421} + 8m^4\mu_{422}g_{10}, \quad S_3 = g_{136}\mu_{441} + g_{06}\mu_{442}$$

$$S_4 = g_{06}\mu_{12} + 8m^4\mu_{11}g_{10}, \quad S_{13} = g_{136} - g_{06}\mu_{11}$$

$$g_T = (\gamma_{24}^2\gamma_{T1} - \gamma_5\gamma_{T2}) + \frac{4}{\pi}\frac{\alpha}{b}\gamma_5(\gamma_{T2} - \gamma_5\gamma_{T1})\varepsilon^{1/2}$$

and in Eqs. (36), (37a) and (37b)

$$C_{11} = g_{T1}/g_T, \quad g_{T1} = \gamma_{24}^2 - \frac{4}{\pi}\frac{\alpha}{b}\gamma_5^2\varepsilon^{1/2}$$

$$\Theta_3 = \frac{1}{C_3}\left\{\gamma_{14}\gamma_{24}\frac{m^4\mu_{11}}{16n^2\beta^2 g_{09}g_{06}}\varepsilon^{-1} - \gamma_{14}\gamma_{24}\frac{m^2 g_{11}}{32n^2\beta^2 g_{09}} + \frac{1}{8}\frac{\gamma_5}{g_{T1}}\left[m^2\left[\mu_{12} - 8(\mu d_{20})^2\right]\varepsilon - 2g_{05}\varepsilon^2 + \frac{g_{05}^2}{m^2}\varepsilon^3\right]\right.$$
$$\left. + \frac{\gamma_{24}^2 - \gamma_5^2}{\gamma_{24}}\frac{\gamma_{T2}}{g_T}\lambda_T^{(2)}\right\}$$

$$\Theta_4 = \left[\frac{\gamma_5}{\gamma_{24}}\frac{P_0}{P_{cr}} + \frac{\gamma_{24}^2 - \gamma_5^2}{\gamma_{24}}\frac{\gamma_{T2}}{g_T}\left(1 - \frac{P_0}{P_{cr}}\right)\right]\lambda_T^{(0)}$$

$$C_3 = 1 - \mu d_{20} - \frac{g_{05}}{m^2}\varepsilon + \frac{\gamma_{24}^2 - \gamma_5^2}{\gamma_{24}}\frac{\gamma_{T2}}{g_T}\lambda_T^{(1)}$$

$$\lambda_T^{(0)} = 2\lambda_p^{(0)}$$

$$\lambda_T^{(1)} = 2\lambda_p^{(1)}$$

$$\lambda_T^{(2)} = 2\lambda_p^{(2)} - \frac{1}{8}\frac{\gamma_{24}}{g_{T1}}\left\{m^2\left[\mu_{12} - 8(\mu d_{20})^2\right]\varepsilon - 2g_{05}\varepsilon^2 + \frac{g_{05}^2}{m^2}\varepsilon^3\right\}$$

$$\lambda_T^{(3)} = 2\lambda_p^{(3)}$$

$$\lambda_T^{(4)} = 2\lambda_p^{(4)} + \frac{1}{64}\frac{\gamma_{24}}{g_{T1}}\left\{\frac{b_{11}}{32\pi\alpha}\gamma_{14}^2\gamma_{24}^2\frac{m^8\mu_{11}^2}{n^4\beta^4 g_{09}^2 g_{06}^2}\varepsilon^{-3/2} + m^2n^4\beta^4\mu_{11}^2\left(\frac{S_4}{S_1}\right)^2\varepsilon^3\right\} \tag{A.2}$$

in the above equations (with g_{ij} and g_{ijk} are defined as in Shen, 2001)

$$b = \left[\frac{\gamma_{14}\gamma_{24}\gamma_{320}^2}{g_{16}}\right]^{1/2}, \quad c = \gamma_{14}\gamma_{24}\gamma_{320}\frac{g_{15}}{2g_{16}}, \quad \alpha = [(b-c)/2]^{1/2}, \quad \phi = [(b+c)/2]^{1/2}$$

$$a_{01}^{(1)} = a_{01}^{(3/2)} = 1, \quad a_{10}^{(1)} = a_{10}^{(3/2)} = \frac{\alpha}{\phi}g_{17}, \quad b_{01}^{(2)} = b_{01}^{(5/2)} = \gamma_{24}g_{19}, \quad b_{10}^{(2)} = b_{10}^{(5/2)} = \gamma_{24}\frac{\alpha}{\phi}g_{20} \tag{A.3}$$

$$b_{11} = \frac{1}{b}\left[\left(a_{10}^{(1)}\right)^2\phi^2 b + a_{10}^{(1)}2\alpha\phi c + (2\alpha^4 - \alpha^2\phi^2 + \phi^4)\right]$$

and

$$\mu_{11} = 1 + \mu d_{11}, \quad \mu_{21} = 2 + \mu d_{11}, \quad \mu_{31} = 3 + \mu d_{11}, \quad \mu_{12} = 1 + 2\mu d_{11}$$
$$\mu_{113} = 1 + \mu d_{11} + \mu d_{13}, \quad \mu_{213} = 1 + \mu_{113}, \quad \mu_{421} = \mu_{12}\mu_{31} + 2\mu_{11}^2, \quad \mu_{422} = \mu_{11}\mu_{21},$$
$$\mu_{301} = \frac{\mu d_{11}}{\mu_{11}^2}, \quad \mu_{302} = \frac{\mu d_{11}}{\mu_{11}}, \quad \mu_{201} = \frac{\mu_{21}}{\mu_{11}}, \quad \mu_{222} = \frac{\mu_{12}\mu_{113}}{\mu_{11}}$$
$$\mu_{420} = 2 - \frac{\mu_{12}}{\mu_{11}^2}, \quad \mu_{441} = \mu_{21} + 2\mu_{113} + \frac{\mu_{113}^2}{\mu_{11}} + \frac{\mu_{113}^3}{\mu_{11}},$$
$$\mu_{442} = \mu_{21}\mu_{31} + 2\mu_{113} - \mu_{113}^2 - \mu_{113}^3$$
$$d_{11} = \frac{8}{\pi^2}\frac{\gamma_{C2}}{(\gamma_{C1}^2 + m^2)(\gamma_{C2}^2 + n^2)}\left[\gamma_{C1} + \left(m\sin\frac{m\pi}{2} - \gamma_{C1}\cos\frac{m\pi}{2}\right)\exp\left(-\frac{\pi\gamma_{C1}}{2}\right)\right][1 - (-1)^n\exp(-\pi\gamma_{C2})]$$
$$d_{13} = \frac{8}{\pi^2}\frac{\gamma_{C2}}{(\gamma_{C1}^2 + m^2)(\gamma_{C2}^2 + 9n^2)}\left[\gamma_{C1} + \left(m\sin\frac{m\pi}{2} - \gamma_{C1}\cos\frac{m\pi}{2}\right)\exp\left(-\frac{\pi\gamma_{C1}}{2}\right)\right][1 - (-1)^n\exp(-\pi\gamma_{C2})]$$
$$d_{20} = \frac{4}{\pi^2}\frac{\gamma_{C1}}{(\gamma_{C1}^2 + 4m^2)\gamma_{C2}}\left[1 - (-1)^n\exp\left(-\frac{\pi\gamma_{C1}}{2}\right)\right][1 - \exp(-\pi\gamma_{C2})]$$
$$d_{02} = \frac{4}{\pi^2}\frac{\gamma_{C2}}{\gamma_{C1}(\gamma_{C2}^2 + 4n^2)}\left[1 - \exp\left(-\frac{\pi\gamma_{C1}}{2}\right)\right][1 - \exp(-\pi\gamma_{C2})]$$
$$d_{04} = \frac{4}{\pi^2}\frac{\gamma_{C2}}{\gamma_{C1}(\gamma_{C2}^2 + 16n^2)}\left[1 - \exp\left(-\frac{\pi\gamma_{C1}}{2}\right)\right][1 - \exp(-\pi\gamma_{C2})] \tag{A.4}$$

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